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Intermittency corrections to spectra of temperature fluctuations in isotropic turbulence

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Abstract. The purpose of this paper is to describe and compare the dissipative effects on the scaling laws of the Obukhov-Corrsin universal equilibrium theory as intermittency corrections formulated according to the log-normal model of Obukhov and Kolmogorov and the β model of Frisch *et al* under conditions of Prandtl numbers away from unity.

1. Introduction

According to the Obukhov-Corrsin theory [1, 2] the small-scale structure of the temperature field has a universal character. That is, in turbulence at high Reynolds and Peclet numbers, there exists for large wavenumbers a convective subrange where turbulence is isotropic and is uniquely determined by the average dissipation rates χ and ε, the kinematic viscosity ν and the thermal diffusivity K, and is independent of the large-scale features of the flow. However, the Obukhov-Corrsin theory has not taken into account the intermittency in the flow that leads to the spatial randomness of dissipation rates χ and ε, which would be expected to depend on Reynolds and Peclet numbers and to cause at the lower end of the convective subrange systematic departures from the Obukhov-Corrsin scaling laws which use mean dissipation values. One could now follow Obukhov [3] and Kolmogorov [4] and reformulate the ideas in the Obukhov-Corrsin theory by introducing local mean dissipations, determined by averaging over a suitably small region in space. Under the assumption that these are random variables following logarithmic-normal distributions, one could then deduce how some of the original local similarity arguments applied to structure functions would change under the new interpretation. Alternatively, one could follow Mandelbrot [5] and argue that the deviations from the Obukhov-Corrsin scaling laws are related to the fractal aspects of the geometry of turbulence. In particular, one may assume that the dissipations χ and ε are concentrated on a set with non-integer Hausdorff dimension. These ideas can be formulated in a simpler way through the so-called β model [6]. The key assumption in the β model is that the flux of energy is transferred to only a fixed fraction β of the eddies downstream in the cascade. A noteworthy feature of this approach is that one does not have to assume the Obukhov-Corrsin scaling law initially and then derive its modified version by somehow mysteriously taking into account fluctuations in the dissipations. The purpose of this paper is to carry out both the log-normal and β-model formulations and to compare and discuss the results under conditions where the Prandtl number is away from unity.

2. Log–normal model

The temperature structure function in the inertial-convective range assumes the form

$$D(r) \sim \chi \varepsilon^{-1/3} r^{2/3} \tag{1}$$

which is the well known $\frac{2}{3}$ power law. In the viscous-convective range, the viscous effects may be imagined to cause spatial randomness of ε, while χ remains invariant in view of the absence of thermal diffusive effects in this range. Following Obukhov [3] and Kolmogorov [4], let us introduce for ε a local mean value ε_r determined by averaging over a suitably small region in space of dimension r. Let us further assume that ε_r is a random variable following a logarithmic normal distribution, so that

$$\langle \varepsilon_r^\alpha \rangle = \bar{\varepsilon}^\alpha \, \mathrm{e}^{\alpha(\alpha-1)\sigma^2/2} \tag{2}$$

where σ^2 is the variance of log ε_r and $\bar{\varepsilon}$ does not depend on r. The variance σ^2 has the form

$$\sigma^2 = Q + \mu \log\left(\frac{L}{r}\right) \tag{3}$$

where L is the integral scale of turbulence, Q is a function depending upon the large scales, and μ is a universal positive constant.

Inserting (3) into (2) yields

$$\langle \varepsilon_r^\alpha \rangle \sim \bar{\varepsilon}^\alpha \left(\frac{L}{r}\right)^{\mu\alpha(\alpha-1)/2} \tag{4}$$

and, in particular,

$$\langle \varepsilon_r^{-1/3} \rangle \sim \bar{\varepsilon}^{-1/3} \left(\frac{r}{L}\right)^{-2\mu/9} \tag{5}$$

Replacing $\varepsilon^{-1/3}$ with $\langle \varepsilon_r^{-1/3} \rangle$ and χ with $\bar{\chi}$ in (1), we obtain

$$D(r) \sim \bar{\chi} \bar{\varepsilon}^{-1/3} r^{2/3} \left(\frac{r}{L}\right)^{-2\mu/9} \tag{6}$$

which show deviations from the $r^{2/3}$ law due to the dissipation fluctuations. The spectral law in the inertial subrange corresponding to (6) is therefore given by

$$\Gamma(k) \sim \bar{\chi} \bar{\varepsilon}^{-1/3} k^{-5/3} (kL)^{2\mu/9}. \tag{7}$$

Observe that the intermittency corrections to the temperature spectrum decrease the $\frac{5}{3}$ exponent. Physically, this is due to the fact that the 'driver' velocity fluctuations weakened by the intermittency effects cause the 'driven' temperature fluctuations to pile up in the wavenumber space. Such a state of affairs is well known to prevail in the viscous-convective range for large Prandtl numbers [7]. The intermittency effects in the latter range can be interpreted as being solely those due to viscosity.

In the inertial-diffusive range the thermal diffusive effects cause spatial randomness of χ while ε now remains invariant in view of the absence of viscous effects in this range. However, since $\langle \chi_r \rangle = \bar{\chi}$, where χ_r is the average thermal dissipation in a small volume of dimension r, there is no intermittency correction in this regime based on the log-normal model. This is inconsistent with the theory of Batchelor *et al* [6] which shows that there is a departure from the $k^{-5/3}$ law in this regime, and may therefore be viewed as a defect in the log-normal model.

It may be mentioned that a theory based on the joint probability distribution of ε_r and χ_r as a bivariate log-normal was given by Van Atta [9]. But this theory is restricted to Prandtl numbers of unity so that it cannot treat the viscous-convective and inertial-diffusive regimes prevalent at Prandtl numbers away from unity.

3. The β-model

Consider a discrete sequence of scales

$$l_n = l_0 p^{-n} \qquad n = 0, 1, 2, \ldots \tag{8}$$

and a discrete sequence of wavenumbers $k_n = l_n^{-1}$. The kinetic energy and scalar variance per unit mass in the nth scale are defined by

$$E_n = \int_{k_n}^{k_{n+1}} E(k)\,\mathrm{d}k \tag{9a}$$

and

$$\Gamma_n = \int_{k_n}^{k_{n+1}} \Gamma(k)\,\mathrm{d}k. \tag{9b}$$

Let us assume that we have a statistically stationary turbulence where kinetic energy and scalar variance are introduced into the fluid at scales $\sim l$ and are then transferred successively to scales $l_1, l_2, \ldots$ until some scale l_d where dissipations are able to compete with non-linear transfers. We now make the assumption that at the nth step, only a fraction $\beta^{\mp 2n}$ (the $-$ and $+$ signs refer to the viscous-convective and inertial-diffusive regimes, respectively) of the total space has an appreciable excitation so that the standard Obukhov-Corrsin phenomenology is valid only in this active region. The kinetic energy and the scalar variance per unit mass in the active region of the nth scale are then given by

$$E_n \sim \beta^n V_n^2 \tag{10a}$$

$$\Gamma_n \sim \beta^{\mp 2n} \theta_n^2 \tag{10b}$$

where V_n and θ_n are the characteristic velocity and temperature, respectively, of the nth scale, and

$$\beta^n = (p^{D-3})^n = \left[\frac{l_n}{l_0}\right]^{3-D}. \tag{11}$$

Here D is the fractal dimension $(D < 3)$ of the region in which the dissipations χ and ε are concentrated. The results in (10) are consistent with the experimentally observed fact [10] that the temperature fluctuations exhibit a higher degree of intermittency than the velocity fluctuations. Equation (11) expresses the fact that the intermittency increases with decrease of scale size.

The rates of transfer of kinetic energy and scalar variance for unit mass from the nth scale to the $(n+1)$th scale are given by

$$\varepsilon_n \sim \frac{E_n}{t_n} \sim \frac{\beta^n V_n^3}{l_n} \tag{12a}$$

$$\chi_n \sim \frac{\Gamma_n}{t_n} \sim \frac{\beta^{\mp 2n} \theta_n^2 V_n}{l_n} \tag{12b}$$

where $t_n \sim l_n/V_n$ is a characteristic time of the nth scales. In the convective subrange, we assume a stationary process in which kinetic energy and scalar variance are introduced at scales $\sim l_0$ and removed at scales $\sim l_d$. Conservation of kinetic energy and scalar variance requires that

$$\varepsilon_n = \bar{\varepsilon} \qquad \chi_n = \bar{\chi} \qquad l_d \leqslant l_n \leqslant l_0 \tag{13}$$

where $\bar{\varepsilon}$ and $\bar{\chi}$ are the mean rates of kinetic energy dissipation and scalar variance dissipation, respectively.

Using (13), equations (10)-(12) then give

$$V_n \sim (\bar{\varepsilon} l_n)^{1/3} \left(\frac{l_n}{l_0}\right)^{-(3-D)/3} \tag{14a}$$

and

$$E_n \sim (\bar{\varepsilon} l_n)^{2/3} \left(\frac{l_n}{l_0}\right)^{(3-D)/3}. \tag{14b}$$

Hence, based on (14b) we find the intermittency correction

$$\langle \varepsilon^{n/3} \rangle \sim \bar{\varepsilon}^{n/3} \left(\frac{l_n}{l_0}\right)^{-(n-3)(3-D)/3}. \tag{15}$$

In the viscous-convective regime χ remains invariant, i.e. $\chi = \bar{\chi}$, and we have from (12) and (14)

$$\Gamma_n \sim \bar{\chi} \langle \varepsilon^{-1/3} \rangle l_n^{2/3} \left(\frac{l_n}{l_0}\right)^{-2(3-D)} \tag{16}$$

where, according to (15),

$$\langle \varepsilon^{-1/3} \rangle \sim \bar{\varepsilon}^{-1/3} \left(\frac{l_n}{l_0}\right)^{4(3-D)/3}. \tag{17}$$

Substituting this result into (16) yields

$$\Gamma_n \sim \bar{\chi} \bar{\varepsilon}^{-1/3} l_n^{2/3} \left(\frac{l_n}{l_0}\right)^{-2(3-D)} \tag{18a}$$

and the corresponding spectral law is

$$\Gamma(k) \sim \bar{\chi} \bar{\varepsilon}^{-1/3} k^{-5/3} (k l_0)^{2(3-D)/3}. \tag{18b}$$

Equation (18b) shows that in the viscous-convective regime, the intermittency correction to the temperature spectrum decreases the $\frac{5}{3}$ exponent, similar to that found based on the log-normal model. Observe that (18b) reduces to Batchelor's [11] result if $D = 2$.

In the inertial-diffusive regime it is ε that remains invariant, i.e. $\varepsilon = \bar{\varepsilon}$, and on noting further that $\langle \chi \rangle = \bar{\chi}$, we have from (12) and (14)

$$\Gamma_n \sim \bar{\chi} \bar{\varepsilon}^{-1/3} l_n^{2/3} \left(\frac{l_n}{l_0}\right)^{2(3-D)} \tag{19a}$$

which leads to the spectrum

$$\Gamma(k) \sim \bar{\chi} \bar{\varepsilon}^{-1/3} k^{-5/3} (k l_0)^{-2(3-D)/3}. \tag{19b}$$

Comparison of (18b) and (19b) shows that the θ spectrum falls off much more steeply in the inertial-diffusive range than it does in the viscous-convective range. Physically, this is due to the fact that the temperature fluctuations are being rushed to higher wavenumbers by the strong velocity fluctuations (which are not yet diminished by viscosity), while being cut in by thermal diffusivity. Observe that (19b) reduces to the result of Batchelor *et al* [8] if $D = 1$ (note that the inertial-diffusive regime is more intermittent than the viscous-convective regime).

4. Discussion

We have just seen that the departures from the Obukhov-Corrsin scaling laws can be described in terms of intermittency corrections through the log-normal as well as the β model. In order to compare the two results, let us recall that the universal constant μ and the fractal dimension D are related by

$$\mu = 3 - D. \tag{20}$$

Using (20), comparison of (7) and (18b) shows that the intermittency correction given by (7) is smaller than that given by (18b) by a factor of 3 so that the log-normal model underestimates the intermittency correction in the viscous-convective regime. On the other hand, the log-normal model misses the intermittency corrections altogether in the inertial-diffusive regime. In general the intermittency corrections obtained above may be too small to allow for an experimental verification at the usual level of energy and temperature spectra.

Finally, we mention that one may generalise the β model to admit the possibility that the region of excitation is instead a non-homogeneous fractal. Thus, in the spirit of Mandelbrot's [12] weighted-curdling model, the contraction factors (i.e. β) may be considered to be independent random variables [13] which can take different values in each scale i at the nth step of the cascade.

References

[1] Obukhov A M 1949 *Izv. Akad. Nauk. USSR, Geoge. i Geofiz.* **13** 58
[2] Corrsin S 1951 *J. Appl. Phys.* **22** 469
[3] Obukhov A M 1962 *J. Fluid Mech.* **13** 77
[4] Kolmogorov A N 1962 *J. Fluid Mech.* **13** 82
[5] Mandelbrot B 1972 *Statistical Models and Turbulence* ed M Rosenblatt and C Van Atta (Berlin: Springer) p 333
[6] Frisch U, Sulem P L and Nelkin M 1978 *J. Fluid Mech.* **87** 719
[7] Hill R J 1978 *J. Fluid Mech.* **88** 541
[8] Batchelor G K, Howells I D and Townsend A A 1959 *J. Fluid Mech.* **5** 134
[9] Van Atta C W 1971 *Phys. Fluids* **14** 1803 (Erratum 1973 *Phys. Fluids* **16** 574)
[10] Williams R M and Paulson C A 1977 *J. Fluid Mech.* **83** 547
[11] Batchelor G K 1959 *J. Fluid Mech.* **5** 113
[12] Mandelbrot B 1976 *Turbulence and Navier-Stokes Equations* ed R Temam (Lecture Notes in Math **565**) (Berlin: Springer) p 121
[13] Benzi R, Paladin G, Parisi G and Vulpiani A 1984 *J. Phys. A: Math. Gen.* **17** 3521